

# A proposed proper EPRL vertex amplitude

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## Abstract

As established in a prior work of the author, the linear simplicity constraints used in the construction of the so-called ‘new’ spin-foam models mix three of the five sectors of Plebanski theory, only one of which is gravity in the usual sense, and this is the reason for certain ‘unwanted’ terms in the asymptotics of the EPRL vertex amplitude as calculated by Barrett et al.

In the present paper, an explicit classical discrete condition is derived that isolates the desired gravitational sector, which we call  $(II+)$ , following other authors. This condition is quantized and used to modify the vertex amplitude, yielding what we call the ‘proper EPRL vertex amplitude.’ This vertex still depends only on standard  $SU(2)$  spin-network data on the boundary, is  $SU(2)$  gauge-invariant, and is linear in the boundary state, as required. In addition, the asymptotics now consist in the single desired term of the form  $e^{iS_{\text{Regge}}}$ , and all degenerate configurations are exponentially suppressed.

## 1 Introduction

At the heart of the path integral formulation of quantum mechanics [1, 2] is the prescription that the contribution to the transition amplitude by each classical trajectory should be the exponential of  $i$  times the classical action. The use of such an expression has roots tracing back to Paul Dirac’s *Principles of Quantum Mechanics* [3], and is central to the successful derivation of the classical limit of the path integral, using the fact that the classical equations of motion are the stationary points of the classical action.

The modern spin-foam program [4–6] aims to provide a definition, via path integral, of the dynamics of loop quantum gravity (LQG) [4, 6–8], a background independent canonical quantization of general relativity. The only spin-foam model to so far match the kinematics of loop quantum gravity and therefore achieve this goal is the so-called EPRL model [9–12], which, for Barbero-Immirzi parameter less than 1 is equal to the FK model [13].

In loop quantum gravity, geometric operators have *discrete* spectra. The basis of states diagonalizing the area and other geometric operators are the *spin-network states*. The spin-foam path integral consists in a sum over amplitudes associated to histories of such states, called *spin-foams*. Each spin-foam in turn can be interpreted in terms of a Regge geometry on a simplicial lattice. The simplest amplitude provided by a spin-foam model is the so-called *vertex amplitude* which gives the probability amplitude for a set of quantum data on the boundary of single 4-simplex.

The semiclassical limit [14] of the EPRL vertex amplitude, however, is *not* equal to the exponential of the Regge action as one would desire, but rather consists in a sum of terms, each of which is the

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exponential of some action.<sup>1</sup> As pointed out in the recent work [16], these terms correspond to three different sectors of Plebanski theory, only one of which — often called the (II+) sector — is the gravitational sector one wishes to include. Furthermore, the reason for the presence of these three sectors was seen to be that the so-called *linear simplicity constraints* — constraints which are also used in the Freidel-Krasnov model [13] — mix precisely these three sectors.

In this paper, we propose a modification to the EPRL vertex amplitude which solves this problem. We begin by deriving, at the classical discrete level, a condition which isolates the (II+) sector. This condition is then appropriately quantized and inserted into the expression for the vertex, leading to a modification of the EPRL vertex amplitude. In this sense we, for the first time, propose a spin-foam vertex amplitude in which restriction to the *single* gravitational sector (II+) is completely imposed. The resulting vertex continues to be a function of a loop quantum gravity boundary state and hence may still be used to define dynamics *for loop quantum gravity*. It furthermore remains linear in the boundary state and fully  $SU(2)$  invariant — two conditions forming a stringent requirement restricting the possible expressions for the vertex. Lastly, as is shown in the final section of this paper, for a complete set of boundary states, the asymptotics of the vertex include only a single term, equal to the exponential of  $i$  times the Regge action. We call the resulting vertex amplitude the *proper EPRL vertex amplitude*.

We begin the paper with a review of the classical discrete framework underlying the spin-foam model and derive the condition isolating the (II+) sector. Then, after briefly reviewing the existing EPRL vertex amplitude, the definition of the new proper vertex is introduced. The last half of the paper is then spent proving the properties summarized above. We then close with a discussion.

## 2 Classical analysis

### 2.1 Background

#### 2.1.1 Generalities

We use the same definitions as in [16]. Let  $\tau^i := \frac{-i}{2}\sigma^i$  ( $i = 1, 2, 3$ ), where  $\sigma^i$  are the Pauli matrices. For each element  $\lambda \in \mathfrak{su}(2)$ ,  $\lambda^i \in \mathbb{R}^3$  shall denote its components with respect to the basis  $\tau^i$ . Let  $I$  denote the  $2 \times 2$  identity matrix. We also freely use the isomorphism between  $\mathfrak{so}(4)$  and  $\mathfrak{spin}(4) := \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ ,  $J^{IJ} \leftrightarrow (J_-, J_+) \equiv (J_- \tau_i, J_+ \tau_i)$  ( $I, J = 0, 1, 2, 3$ ), explicitly given by

$$J_{\pm}^i = \frac{1}{4}\epsilon^i{}_{jk}J^{jk} \pm \frac{1}{2}J^{0i}. \quad (2.1)$$

with inverse

$$\begin{aligned} J^{ij} &= \epsilon^{ij}{}_k (J_+^k + J_-^k) \\ J^{0i} &= J_+^i - J_-^i. \end{aligned} \quad (2.2)$$

Furthermore, we remind the reader [14] of the explicit expression for the action of a  $Spin(4) = SU(2) \times SU(2)$  group elements on  $\mathbb{R}^4$ . For each  $V^I \in \mathbb{R}^4$  define

$$\zeta(V) := V^0 I + i\sigma_i V^i. \quad (2.3)$$

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<sup>1</sup>From [15], this is true also for the Freidel-Krasnov model, as must be the case as it is equal to EPRL for  $\gamma < 1$ . In [15], one finds two terms, not one, in the asymptotics. Furthermore, the presence of only two terms is likely due to their reformulating the model as a discrete first order path integral and then imposing non-degeneracy, a procedure whose equivalent in the spin-foam language, needed for contact with canonical states, is not known [15].

Then the action of  $G = (X^-, X^+)$  is given by

$$\zeta(G \cdot V) = X^- \zeta(V) (X^+)^{-1}. \quad (2.4)$$

### 2.1.2 Discrete classical framework

Spin-foam models of quantum gravity are based on a formulation of gravity as a *constrained BF theory*, using the ideas of Plebanski [17]. In the continuum, the basic variables are an  $\mathfrak{so}(4)$  connection  $\omega_\mu^{IJ}$  and an  $\mathfrak{so}(4)$ -valued two-form  $B_{\mu\nu}^{IJ}$ , which we call the *Plebanski two-form*. The action is

$$S = \frac{1}{2\kappa} \int (B + \frac{1}{\gamma} {}^*B)_{IJ} \wedge F^{IJ}. \quad (2.5)$$

with  $F := d\omega + \omega \wedge \omega$  the curvature of  $\omega$ ,  $\kappa := 8\pi G$ , and  $\gamma \in \mathbb{R}^+$  the Barbero-Immirzi parameter. If  $B_{\mu\nu}^{IJ}$  satisfies what we call the Plebanski constraint [18, 19], it must be one of the five forms

$$(\text{I}\pm) \quad B^{IJ} = \pm e^I \wedge e^J \text{ for some } e_\mu^I$$

$$(\text{II}\pm) \quad B^{IJ} = \pm \frac{1}{2} \epsilon^{IJ}{}_{KL} e^K \wedge e^L \text{ for some } e_\mu^I$$

$$(\text{deg}) \quad \epsilon_{IJKL} \eta^{\mu\nu\rho\sigma} B_{\mu\nu}^{IJ} B_{\rho\sigma}^{KL} = 0 \text{ (degenerate case)}$$

which we call *Plebanski sectors*, and refer to as (I $\pm$ ), (II $\pm$ ), and (deg). Here  $\eta^{\alpha\beta\gamma\delta}$  denotes the Levi-Civita tensor of density weight 1. Only Plebanski sector (II $+$ ) describes gravity in the usual sense, and in this sector the BF action reduces to the *Holst action* for gravity [20],

$$S_{Holst} = \frac{1}{4\kappa} \int \left( \epsilon_{IJKL} e^K \wedge e^L + \frac{2}{\gamma} e_I \wedge e_J \right) \wedge F^{IJ} \quad (2.6)$$

the Legendre transform of which forms the starting point for loop quantum gravity [7, 20].

In spin-foam quantization, one usually introduces a simplicial discretization of space-time. However, in this paper we concern ourselves with the so-called ‘vertex amplitude’, which may be thought of as the transition amplitude for a single 4-simplex. For clarity, we thus focus on a single oriented 4-simplex  $S$ . The EPRL model has also been generalized to general cell-complexes [12]; however because we use the work [14], and because we introduce formulae that, so far, apply only to 4-simplices, we restrict the discussion to the case of a 4-simplex. In  $S$ , number the tetrahedra  $a = 0, \dots, 4$ , and label each triangle  $\Delta_{ab}$  by the pair  $(ab)$  of tetrahedra that contain it. One thinks of each tetrahedron, as well as the 4-simplex itself, as having its own ‘frame’ [10]. One has a parallel transport map from each tetrahedron to the 4-simplex frame, yielding in our case 5 parallel transport maps  $G_a = (X_a^-, X_a^+) \in Spin(4)$ ,  $a = 0, \dots, 4$ . The continuum two-form  $B$  is then represented by the algebra elements  $B_{ab} = \int_{\Delta_{ab}} B$ , where each element is treated as being ‘in the frame at  $a$ .’ For each  $ab$ , in terms of self-dual and anti-self-dual parts, these elements are related to the momenta conjugate to the parallel transports (see section 3.2) by [9, 16]

$$(J_{ab}^\pm)^i = \left( \frac{\gamma \pm 1}{\kappa\gamma} \right) (B_{ab}^\pm)^i. \quad (2.7)$$

We call  $B_{ab}$  and  $J_{ab}$  the *canonical bivectors* due to their role in the canonical theory in section 3.2.

To reconstruct information about the 4-dimensional geometry of the simplex, one must parallel transport all of the bivectors  $B_{ab}^{IJ}$  to a *common* frame, for which purpose we choose the 4-simplex frame, so that we have  $B_{ab}(S) := G_a \triangleright B_{ab}$ , where  $\triangleright$  here and throughout this paper denotes the adjoint action.

### 2.1.3 Linear simplicity

The linear simplicity constraint requires that for each  $a$ , there exist  $N_a^I$  such that

$$C_{ab}^I := N_{aJ} ({}^*B_{ab})^{JI} \approx 0 \quad \forall b \neq a. \quad (2.8)$$

This is a condition on the bivectors in the *4-simplex frame*. The canonical variables, by contrast, are defined in the tetrahedron frames. For each tetrahedron  $a$ , one imposes a gauge-fixed version of (2.8) in which  $N_a^I$  is fixed to be  $\mathcal{N}^I := (1, 0, 0, 0)$  [9], reducing the constraint to

$$C_{ab}^i := \frac{1}{2} \epsilon^i{}_{jk} B_{ab}^{jk} \approx 0. \quad (2.9)$$

The set of canonical bivectors  $B_{ab}^{IJ}$  satisfying these constraints is parameterized by what we call the *reduced boundary data* — one unit 3-vector  $n_{ab}^i$  per ordered pair  $ab$ , and one area  $A_{ab}$  per triangle  $(ab)$  — via

$$B_{ab} = \frac{1}{2} A_{ab} (-n_{ab}, n_{ab}). \quad (2.10)$$

From (2.7) and (2.10), the generators of internal spatial rotations in terms of the reduced boundary data are

$$L_{ab}^i = (J^-)^i{}_{ab} + (J^+)^i{}_{ab} = \frac{1}{\kappa\gamma} A_{ab} n_{ab}^i. \quad (2.11)$$

The bivectors in the 4-simplex frame take the form

$$B_{ab}(S) = B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, X_a^\pm) := \frac{1}{2} A_{ab} (-X_a^- \triangleright n_{ab}, X_a^+ \triangleright n_{ab}). \quad (2.12)$$

We call (2.12) the ‘physical’ bivectors reconstructed from  $A_{ab}, n_{ab}, X_a^\pm$ .

### 2.1.4 Reconstruction theorem

Let  $M$  denote  $\mathbb{R}^4$  as an oriented manifold, equipped with the canonical  $\mathbb{R}^4$  metric. A *geometrical simplex* in  $M$  is the convex hull of 5 points, called vertices, in  $M$ , not all of which lie in the same 3-plane. We define an *ordered 4-simplex*  $\sigma$  to be a geometrical simplex with vertices numbered  $0, \dots, 4$ , such that the ordered set of vectors  $(\vec{01}, \vec{02}, \vec{03}, \vec{04})$  has positive orientation. Each tetrahedron is then labeled by the number of the one vertex it does not contain. Given an ordered 4-simplex in  $M$ , the associated *geometrical bivectors*  $(B_{ab}^{\text{geom}})^{IJ}$  are defined as  $(B_{ab}^{\text{geom}})^{IJ} := A(\Delta_{ab}) \frac{(N_a \wedge N_b)^{IJ}}{|N_a \wedge N_b|}$ , where  $A(\Delta_{ab})$  is the area of  $\Delta_{ab}$ ,  $N_a^I$  is the outward unit normal to tetrahedron  $a$ ,  $(N_a \wedge N_b)^{IJ} := 2N_a^{[I} N_b^{J]}$ , and  $|X^{IJ}|^2 := \frac{1}{2} X^{IJ} X_{IJ}$ .

The following is a partial version of theorem 3 in [14], the same one appearing in [16]. A set of reduced boundary data  $\{A_{ab}, n_{ab}\}$  (1.) is *non-degenerate* if, for each  $a$ , every set of three vectors  $n_{ab}$  with  $b \neq a$  is linearly independent, and (2.) satisfies *closure* if  $\sum_{b \neq a} A_{ab} n_{ab} = 0$ . A set  $\{X_a^\pm\} \subset SU(2)$  satisfies the *orientation constraint* if  $X_a^\pm \triangleright n_{ab} = -X_b^\pm \triangleright n_{ba}$ . Lastly, we call two sets of  $SU(2)$  group elements  $\{U_a^1\}, \{U_a^2\}$  *equivalent*,  $\{U_a^1\} \sim \{U_a^2\}$ , if  $\exists Y \in SU(2)$  and five signs  $\epsilon_a$  such that

$$U_a^2 = \epsilon_a Y U_a^1. \quad (2.13)$$

For the proof of the following, see [14, 16].

**Theorem 1** (Partial version of the reconstruction theorem). *Let a set of non-degenerate reduced boundary data  $\{A_{ab}, n_{ab}\}$  satisfying closure be given, as well as a set  $\{X_a^\pm\} \subset SU(2)$ ,  $a = 0, \dots, 4$ , solving the orientation constraint, such that  $\{X_a^-\} \not\sim \{X_a^+\}$ . Then there exists an ordered 4-simplex  $\sigma$  in  $\mathbb{R}^4$ , unique up to inversion and translation, such that*

$$B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, X_a^\pm) = \mu B_{ab}^{\text{geom}}(\sigma) \quad (2.14)$$

for some  $\mu = \pm 1$ , with  $\mu$  independent of the ambiguity in  $\sigma$ .

## 2.2 Explicit classical expression for the geometrical bivectors, and the restriction to (II+)

We now come to the new part of the classical analysis.

**Lemma 1.** *Let  $\{A_{ab}, n_{ab}, X_a^\pm\}$  be given satisfying the hypotheses of theorem 1 and let  $\sigma$  be the ordered 4-simplex gauranteed to exist by this theorem. Let  $\{N_a^I\}$  denote the outward pointing normals to the tetrahedra of  $\sigma$ . Then*

$$N_a^I = \alpha_a (G_a \cdot \mathcal{N})^I \quad (2.15)$$

for some set of signs  $\alpha_a$ .

**Proof.** We first note that

$$\begin{aligned} (N_a \wedge N_b)^{IJ} (G_a \cdot \mathcal{N})_J &\propto B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, X_{ab}^\pm)^{IJ} (G_a \mathcal{N})_J \\ &\propto [G_a \triangleright (-n_{ab}, n_{ab})]^{IJ} (G_a \mathcal{N})_J \\ &= (G_a)^I{}_K (G_a)^J{}_L (-n_{ab}, n_{ab})^{KL} (G_a)_{IM} \mathcal{N}^M \\ &= (G_a)^J{}_L (-n_{ab}, n_{ab})^{KL} \mathcal{N}_K = (G_a)^J{}_L (-n_{ab}, n_{ab})^{0L} = 0 \end{aligned}$$

where (2.14) was used in the first line, and (2.2) was used in the last line. Since this holds for all  $b$ , it follows that  $G_a \mathcal{N}$  is proportional to  $N_a$ ; as both of these vectors are unit, the the coefficient of proportionality must be  $\pm 1$  for each  $a$ .  $\blacksquare$

For the following theorem and throughout the rest of the paper, let  $\hat{=}$  denote equality modulo multiplication by a *positive* real number.

**Theorem 2.** *Let  $\{A_{ab}, n_{ab}, X_a^\pm\}$  be given satisfying the hypotheses of theorem 1 and let  $\sigma$  be the ordered 4-simplex gauranteed to exist by this theorem. Then*

$$B_{ab}^{\text{geom}}(\sigma) \hat{=} \beta_{ab}(\{G_{a'b'}\}) (G_a \cdot \mathcal{N}) \wedge (G_b \cdot \mathcal{N}) \quad (2.16)$$

where

$$\beta_{ab}(\{G_{a'b'}\}) := -\text{sgn} [\epsilon_{ijk} (G_{ac} \cdot \mathcal{N})^i (G_{ad} \cdot \mathcal{N})^j (G_{ae} \cdot \mathcal{N})^k \epsilon_{lmn} (G_{bc} \cdot \mathcal{N})^l (G_{bd} \cdot \mathcal{N})^m (G_{be} \cdot \mathcal{N})^n] \quad (2.17)$$

with  $\{c, d, e\} = \{0, \dots, 4\} \setminus \{a, b\}$  in any order, and  $\text{sgn}$  is defined to be zero when its argument is zero.

**Proof.** Let  $\{N_a^I\}$  be the outward pointing normals to the tetrahedra of  $\sigma$ . Then they satisfy the four-dimensional closure relation (see appendix A)

$$\sum_a V_a N_a^I = 0 \quad (2.18)$$

where  $V_a > 0$  is the volume of the  $a$ th tetrahedron, implying

$$N_a^I = \frac{1}{V_a} \sum_{a' \neq a} V_{a'} N_{a'}^I. \quad (2.19)$$

Thus

$$\begin{aligned} 0 &< \epsilon(N_a, N_c, N_d, N_e)^2 = -\frac{V_b}{V_a} \epsilon(N_b, N_c, N_d, N_e) \epsilon(N_a, N_c, N_d, N_e) \\ &\hat{=} -\alpha_a \alpha_b \epsilon(G_b \cdot \mathcal{N}, G_c \cdot \mathcal{N}, G_d \cdot \mathcal{N}, G_e \cdot \mathcal{N}) \epsilon(G_a \cdot \mathcal{N}, G_c \cdot \mathcal{N}, G_d \cdot \mathcal{N}, G_e \cdot \mathcal{N}) \\ &= -\alpha_a \alpha_b \epsilon(\mathcal{N}, G_{bc} \cdot \mathcal{N}, G_{bd} \cdot \mathcal{N}, G_{be} \cdot \mathcal{N}) \epsilon(\mathcal{N}, G_{ac} \cdot \mathcal{N}, G_{ad} \cdot \mathcal{N}, G_{ae} \cdot \mathcal{N}) \\ &= -\alpha_a \alpha_b \epsilon_{ijk} (G_{bc} \cdot \mathcal{N})^i (G_{bd} \cdot \mathcal{N})^j (G_{be} \cdot \mathcal{N})^k \epsilon_{lmn} (G_{ac} \cdot \mathcal{N})^l (G_{ad} \cdot \mathcal{N})^m (G_{ae} \cdot \mathcal{N})^n \end{aligned}$$

where  $\{\alpha_a\}$  are the signs in lemma 1. Therefore

$$\beta_{ab}(\{G_{a'b'}\}) = \alpha_a \alpha_b \quad (2.20)$$

where  $\beta_{ab}(\{G_{a'b'}\})$  is as in (2.17). We thus have

$$B_{ab}^{\text{geom}}(\sigma) \hat{=} N_a \wedge N_b = \alpha_a \alpha_b (G_a \cdot \mathcal{N}) \wedge (G_b \cdot \mathcal{N}) = \beta_{ab}(\{G_{a'b'}\}) (G_a \cdot \mathcal{N}) \wedge (G_b \cdot \mathcal{N}).$$

■

Throughout this paper, let  $\beta_{ab}(\{G_{a'b'}\})$  be defined by (2.17), and for convenience we define  $\tilde{B}_{ab}^{\text{geom}}(G_a) := \beta_{ab}(\{G_{a'b'}\}) (G_a \cdot \mathcal{N}) \wedge (G_b \cdot \mathcal{N})$ , the right hand side of (2.16).

Because the expression  $(G \cdot \mathcal{N})^i$  used above will appear often, it is useful to stop for a moment to prove some facts about it. From (2.3) and (2.4),

$$(G_{ab} \mathcal{N})^0 I + i \sigma_i (G_{ab} \cdot \mathcal{N})^i = \zeta(G_{ab} \cdot \mathcal{N}) = X_{ab}^- X_{ba}^+,$$

from which one obtains the alternate expression

$$(G_{ab} \cdot \mathcal{N})^i = \text{tr}(\tau^i X_{ab}^- X_{ba}^+). \quad (2.21)$$

The meaning of this latter expression in turn is made clear in the following definition.

**Definition 1.** Given  $g \in SU(2)$  not equal to  $\pm I$ , there exists a unique unit vector  $n[g]^i \in \mathbb{R}^3$  and  $\alpha[g] \in (0, 2\pi)$  satisfying

$$g = \exp(\alpha[g] \cdot n[g] \cdot \tau) = \cos\left(\frac{\alpha[g]}{2}\right) + i n[g] \cdot \sigma \sin\left(\frac{\alpha[g]}{2}\right). \quad (2.22)$$

We call  $n[g]^i$  the proper axis of  $g$ .

In terms of the above definition, one has

$$(G_{ab} \cdot \mathcal{N})^i = \text{tr}(\tau_i X_{ab}^- X_{ba}^+) = \sin\left(\frac{\alpha[X_{ab}^- X_{ba}^+]}{2}\right) n[X_{ab}^- X_{ba}^+]^i. \quad (2.23)$$

**Lemma 2.** Let  $\{A_{ab}, n_{ab}, X_a^\pm\}$  be given satisfying the hypotheses of theorem 1 and let  $\sigma$  be the ordered 4-simplex thereby guaranteed to exist. Then

$$\mu = B_{ab}^{\text{geom}}(\sigma)_{IJ} B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, G_a)^{IJ} \hat{=} \beta_{ab}(\{G_{a'b'}\}) \text{tr}(\tau_i X_{ab}^- X_{ba}^+) L_{ab}^i. \quad (2.24)$$

**Proof.** Starting from (2.14) and theorem 2,

$$\begin{aligned} \mu &= B_{ab}^{\text{geom}}(\sigma)_{IJ} B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, G_a)^{IJ} \\ &\hat{=} \beta_{ab}(\{G_{a'b'}\}) [(G_a \cdot \mathcal{N}) \wedge (G_b \cdot \mathcal{N})]_{IJ} \frac{1}{2} A_{ab} [G_a \triangleright (-n_{ab}, n_{ab})]^{IJ} \\ &= \frac{1}{2} A_{ab} \beta_{ab}(\{G_{a'b'}\}) [\mathcal{N} \wedge (G_{ab} \cdot \mathcal{N})]_{IJ} [-n_{ab}, n_{ab}]^{IJ} \\ &= A_{ab} \beta_{ab}(\{G_{a'b'}\}) [\mathcal{N} \wedge (G_{ab} \cdot \mathcal{N})]_{0i} [-n_{ab}, n_{ab}]^{0i} \\ &= 2 A_{ab} \beta_{ab}(\{G_{a'b'}\}) (G_{ab} \cdot \mathcal{N})_i n_{ab}^i \hat{=} \beta_{ab}(\{G_{a'b'}\}) (G_{ab} \cdot \mathcal{N})_i L_{ab}^i \\ &= \beta_{ab}(\{G_{a'b'}\}) \text{tr}(\tau_i X_{ab}^- X_{ba}^+) L_{ab}^i. \end{aligned}$$

■

We now come to the classical condition isolating the (II+) sector.

**Theorem 3.** *Let a set of non-degenerate reduced boundary data  $\{A_{ab}, n_{ab}\}$  satisfying closure be given, as well as a set  $\{X_a^\pm\} \subset SU(2)$ ,  $a = 0, \dots, 4$  solving the orientation constraint. Then  $B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, G_a)$  is in Plebanski sector (II+) in the sense of [16] iff*

$$\beta_{ab}(\{G_{a'b'}\}) \text{tr}(\tau_i X_{ab}^- X_{ba}^+) L_{ab}^i > 0 \quad (2.25)$$

for any one pair  $a, b$ .

**Proof.**

( $\Rightarrow$ ) Suppose  $B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, G_a)^{IJ}$  is in Plebanski sector (II+) in the sense of [16]. Then by theorem 3 in [16],  $\{X_a^-\} \not\sim \{X_a^+\}$ , so that  $\mu$  exists, and  $\mu = 1$ . Lemma 2 then implies (2.25).

( $\Leftarrow$ ) Suppose (2.25) holds. Suppose by way of contradiction  $\{X_a^-\} \sim \{X_a^+\}$ . Then  $\text{tr}(\tau^i X_{ab}^- X_{ba}^+) = 0$  contradicting (2.25). Therefore  $\{X_a^-\} \not\sim \{X_a^+\}$ . Lemma 2 together with (2.25) then implies  $\mu = +1$ , so that theorem 3 in [16] implies  $B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, G_a)^{IJ}$  is in Plebanski sector (II+). ■

### 3 Review of quantum framework and the EPRL vertex

#### 3.1 Notation for $SU(2)$ and $Spin(4)$ structures.

Let  $V_j$  denote the carrying space for the spin  $j$  representation of  $SU(2)$ , and  $\rho_j(g), \rho_j(x)$  the representation of  $g \in SU(2)$  and  $x \in \mathfrak{su}(2)$  thereon, with the  $j$  subscript dropped when it is clear from the context. Let  $\hat{L}^i := i\rho(\tau^i)$  denote the generators in each of these representation according to the context. Let  $\epsilon : V_j \times V_j \rightarrow \mathbb{C}$  denote the skew-symmetric bilinear epsilon inner product, and  $\langle \cdot, \cdot \rangle$  the hermitian inner product, on  $V_j$  [4, 14]. These inner products determine an antilinear structure map  $J : V_j \rightarrow V_j$  by  $\langle \psi, \phi \rangle = \epsilon(J\psi, \phi)$ .  $J$  commutes with all group representation matrices, which implies that it anti-commutes with all generators.

Let  $V_{j^-, j^+} = V_{j^-} \otimes V_{j^+}$  denote the carrying space for the spin  $(j^-, j^+)$  representation of  $Spin(4) \equiv SU(2) \times SU(2)$ , and  $\rho_{j^-, j^+}(X^-, X^+) := \rho_{j^-}(X^-) \otimes \rho_{j^+}(X^+)$  the representation of  $(X^-, X^+) \in Spin(4)$  thereon, again with the subscript dropped when it is clear from the context.  $\hat{J}_-^i := i\rho(\tau^i) \otimes I_{j^+}$  and  $\hat{J}_+^i := iI_{j^-} \otimes \rho(\tau^i)$  are then the anti-self-dual and self-dual generators respectively, so that  $\hat{L}^i := \hat{J}_-^i + \hat{J}_+^i$  are the generators of spatial rotations on  $V_{j^-, j^+}$ . Define the bilinear form  $\epsilon : V_{j^+, j^-} \times V_{j^+, j^-} \rightarrow \mathbb{C}$  by  $\epsilon(\psi^+ \otimes \psi^-, \phi^+ \otimes \phi^-) := \epsilon(\psi^+, \phi^+) \epsilon(\psi^-, \phi^-)$ , and the antilinear map  $J : V_{j^-, j^+} \rightarrow V_{j^-, j^+}$  by  $J : \psi^+ \otimes \psi^- \mapsto (J\psi^+) \otimes (J\psi^-)$ , so that

$$\langle \Psi, \Phi \rangle = \epsilon(J\Psi, \Phi). \quad (3.1)$$

As in the case of the  $SU(2)$  representations, all  $Spin(4)$  representation operators commute with  $J$ , and all generators anticommute with  $J$ . Lastly, let  $\iota_k^{j^-, j^+}$  denote the intertwining map from  $V_k$  to  $V_{j^-} \otimes V_{j^+}$ , scaled such that it is isometric in the Hilbert space inner products.

#### 3.2 Canonical phase space, kinematical quantization, and the EPRL vertex

In the general boundary formulation of quantum mechanics [4], one associates to the boundary of any 4-dimensional region a *phase space*, whose quantization yields the *boundary Hilbert space* of the theory for that region. In the present case, the region is the 4-simplex  $S$ . The boundary data consists in the algebra elements  $B_{ab}$  and  $J_{ab}$  in the frame of each tetrahedron  $a$ , and for each pair of tetrahedra  $a, b$  one has a parallel transport map  $G_{ab}$  from  $b$  to  $a$ , related to the  $G_a$  introduced in section 2.1.2 by  $G_{ab} = (G_a)^{-1} G_b$ . These boundary data form a classical phase space isomorphic to

the cotangent bundle over any choice of five independent parallel transport maps  $G_{ab} = (X_{ab}^+, X_{ab}^-)$ ,  $\Gamma = T^*(Spin(4)^5) = T^*((SU(2) \times SU(2))^5)$ , which for simplicity we choose to be the ones with  $a < b$ . For  $a < b$ ,  $J_{ab} = (J_{ab}^-, J_{ab}^+)$  and  $J_{ba} = (J_{ba}^-, J_{ba}^+)$  respectively generate right and left translations on  $G_{ab}$ .

The boundary Hilbert space of states  $\mathcal{H}_{\partial S}^{Spin(4)}$  is the  $L^2$  space over the five  $G_{ab} = (X_{ab}^-, X_{ab}^+) \in Spin(4)$  with  $a < b$ . The momenta operators  $\hat{J}_{ab}^\pm$  and  $\hat{J}_{ba}^\pm$  then act by  $i$  times right and left invariant vector fields, respectively, on the elements  $X_{ab}^\pm$ , and, in terms of these,  $\hat{L}_{ab}^i := (\hat{J}_{ab}^-)^i + (\hat{J}_{ab}^+)^i$ . One can define an overcomplete basis of  $\mathcal{H}_{\partial S}^{Spin(4)}$ , the *projected spin-network states* (see [21, 22]), each element of which is labeled by four spins  $j_{ab}^\pm, k_{ab}, k_{ba}$  and two states  $\psi_{ab} \in V_{k_{ab}}, \psi_{ba} \in V_{k_{ba}}$  per triangle:

$$\Psi_{\{j_{ab}^\pm, k_{ab}, \psi_{ab}\}}(X_{ab}^-, X_{ab}^+) := \prod_{a < b} \epsilon(l_{k_{ab}}^{j_{ab}^-, j_{ab}^+} \psi_{ab}, \rho(X_{ab}^-, X_{ab}^+) l_{k_{ba}}^{j_{ab}^-, j_{ab}^+} \psi_{ba}). \quad (3.2)$$

When acting on such a state, the operators  $\hat{L}_{ab}^i, \hat{L}_{ba}^i$  act specifically on the irreducible representation (irrep) vectors  $\psi_{ab}, \psi_{ba}$ :

$$\hat{L}_{ab}^i \Psi_{\{j_{cd}^\pm, k_{cd}, \psi_{cd}\}} = \epsilon(l_{k_{ab}}^{j_{ab}^-, j_{ab}^+} \hat{L}^i \psi_{ab}, \rho(G_{ab}) l_{k_{ba}}^{j_{ab}^-, j_{ab}^+} \psi_{ba}) \prod_{c < d, (cd) \neq (ab)} \epsilon(l_{k_{cd}}^{j_{cd}^-, j_{cd}^+} \psi_{cd}, \rho(G_{cd}) l_{k_{dc}}^{j_{cd}^-, j_{cd}^+} \psi_{dc}), \quad (3.3)$$

$$\hat{L}_{ba}^i \Psi_{\{j_{cd}^\pm, k_{cd}, \psi_{cd}\}} = \epsilon(l_{k_{ab}}^{j_{ab}^-, j_{ab}^+} \psi_{ab}, \rho(G_{ab}) l_{k_{ba}}^{j_{ab}^-, j_{ab}^+} \hat{L}^i \psi_{ba}) \prod_{c < d, (cd) \neq (ab)} \epsilon(l_{k_{cd}}^{j_{cd}^-, j_{cd}^+} \psi_{cd}, \rho(G_{cd}) l_{k_{dc}}^{j_{cd}^-, j_{cd}^+} \psi_{dc}). \quad (3.4)$$

In terms of the projected spin-network over-complete basis, the linear simplicity constraint, when quantized as in [9], is equivalent to

$$k_{ab} = \frac{2j_{ab}^-}{|1 - \gamma|} = \frac{2j_{ab}^+}{|1 + \gamma|} = k_{ba} \quad (3.5)$$

for all  $a \neq b$ . The projected spin networks satisfying linear simplicity are thus parameterized by one spin  $k_{ab}$  and two states  $\psi_{ab}, \psi_{ba} \in V_{k_{ab}}$  per triangle  $(ab)$ , the same parameters specifying a generalized  $SU(2)$  *spin-network state* of LQG:

$$\Psi_{\{k_{ab}, \psi_{ab}\}}(X_{ab}) := \prod_{a < b} \epsilon(\psi_{ab}, \rho(X_{ab}) \psi_{ba}) \in \mathcal{H}_{\partial S}^{LQG} \equiv L^2(SU(2)^{10}). \quad (3.6)$$

Because  $j_{ab}^\pm = \frac{1}{2}|1 \pm \gamma|k_{ab}$  are always half-integers, one deduces that only certain values of the spins  $k_{ab}$  are allowed; let  $\mathcal{K}_\gamma$  be this set of allowable values, and let  $\mathcal{H}_{\partial S}^\gamma$  be the span of the  $SU(2)$  spin-networks (3.6) with  $\{k_{ab}\} \subset \mathcal{K}_\gamma$ . One has an embedding

$$\begin{aligned} \iota : \mathcal{H}_{\partial S}^\gamma &\rightarrow \mathcal{H}_{\partial S}^{Spin(4)} \\ \Psi_{\{k_{ab}, \psi_{ab}\}} &\mapsto \Psi_{\{s_{ab}^\pm, k_{ab}, \psi_{ab}\}} \end{aligned} \quad (3.7)$$

where here, and throughout the rest of the paper, we set

$$s^\pm := \frac{1}{2}|1 \pm \gamma|k. \quad (3.8)$$

Due to (3.3) and (3.4) (and because the  $SU(2)$  spin-networks satisfy a similar property), this embedding in fact *intertwines* the spatial rotation generators  $\hat{L}_{ab}^i$  in the  $Spin(4)$  and  $SU(2)$  theories. Through the embedding  $\iota$ , the operators  $\hat{L}_{ab}^i$  in the  $SU(2)$  theory thus have the same physical meaning as the corresponding operators in the  $Spin(4)$  boundary theory. *Side remark.* One can now ask the question: Is this resulting physical meaning of  $\hat{L}_{ab}^i$  in the  $SU(2)$  boundary theory the same as that



of the *LQG flux operators*  $\hat{E}(\Delta_{ab})^i$ ? The answer is: only if one is in Plebanski sector (II+). In the (II+) sector classically one has  $\kappa\gamma L_{ab}^i := 2B_{ab}^{0i} = \int_{\Delta_{ab}} \epsilon^i_{jk} e^j \wedge e^k =: E(\Delta_{ab})^i$ . In the (II-) sector, this equation differs by a minus sign, and in the degenerate sector,  $e^I$  and hence  $e^i$  does not exist, so that  $E(\Delta_{ab})^i$  doesn't even make sense. This provides yet another motivation to impose a restriction to (II+): It is only then that one can unambiguously interpret the final  $SU(2)$  boundary operators as those of LQG.

Having reviewed the above, the EPRL vertex for a given LQG boundary state  $\Psi_{\{k_{ab}, \psi_{ab}\}}^{LQG} \in \mathcal{H}_{\partial S}^\gamma \subset \mathcal{H}_{\partial S}^{LQG}$  is then

$$\begin{aligned} A_v(\{k_{ab}, \psi_{ab}\}) &:= A_v(\Psi_{\{k_{ab}, \psi_{ab}\}}) = \int_{\text{Spin}(4)^5} \prod_a dG_a (\iota \Psi_{\{k_{ab}, \psi_{ab}\}})(G_{ab}) \\ &= \int_{\text{Spin}(4)^5} \prod_a dG_a \prod_{a < b} \epsilon(l_{k_{ab}}^{j_{ab}^-, j_{ab}^+} \psi_{ab}, \rho(G_{ab}) l_{k_{ab}}^{j_{ab}^-, j_{ab}^+} \psi_{ba}). \end{aligned} \quad (3.9)$$

## 4 Proposed proper EPRL vertex

### 4.1 Definition

Let us consider the structure of the original EPRL vertex amplitude (3.9): The integration over the group elements  $G_a$  can, in a precise sense, be interpreted as a “sum over histories” of parallel transports from the tetrahedra frames to the 4-simplex frames. This integration over the  $G_a$ ’s inside the vertex amplitude can be thought of as a remnant of the process of integrating out the discrete connection used to obtain the initial BF spin-foam model (see [23]). Furthermore, in the semiclassical analysis [14], one sees that the  $G_a$ ’s over which one integrates in (3.9) play precisely the role of such parallel transports. Given this interpretation of the  $G_a$ ’s, in order to impose the desired restriction to Plebanski sector (II+), one must restrict the discrete history data  $G_a$  so that they satisfy the inequality (2.25):

$$\beta_{ab}(\{G_{a'b'}\}) \text{tr}(\tau_i X_{ab}^- X_{ba}^+) L_{ab}^i > 0. \quad (4.1)$$

Normally one would do this by inserting into the path integral

$$\Theta(\beta_{ab}(\{G_{a'b'}\}) \text{tr}(\tau_i X_{ab}^- X_{ba}^+) L_{ab}^i) \quad (4.2)$$

where  $\Theta$  is the Heaviside function, defined to be zero when its argument is zero. However, in the integral (3.9), it is not the classical quantity  $L_{ab}^i$  that appears, but rather *states*  $\psi_{ab}$  in irreducible representations of the corresponding operators  $\hat{L}_{ab}^i$ .<sup>2</sup> As noted in equations (3.3) and (3.4),  $\hat{L}_{ab}^i$  acts on  $\psi_{ab}$  via the  $SU(2)$  generators  $\hat{L}^i$ . Therefore, we partially ‘quantize’ the expression (4.2) by replacing  $L_{ab}^i$  with the generators  $\hat{L}^i$ , yielding the following  $G_a$ -dependent operator acting in the spin  $k_{ab}$  representation of  $SU(2)$ :

$$P_{ba}(\{G_{a'b'}\}) := P_{(0,\infty)} \left( \beta_{ab}(\{G_{a'b'}\}) \text{tr}(\tau_i X_{ba}^- X_{ab}^+) \hat{L}^i \right), \quad (4.3)$$

---

<sup>2</sup>If one uses coherent boundary data as will be done in the next section, then one does have a classical label  $L_{ab}^i$  present, but one would still not be able to simply insert the factor (4.2), as, due to the overcompleteness of the set of coherent states, this would lead to a vertex amplitude that is not linear in the boundary state, something necessary to ensure the final transition amplitude defined by the spin-foam sum is linear in the boundary state.

where  $P_S(\hat{O})$  denotes the spectral projector onto the portion  $S \subset \mathbb{R}$  of the spectrum of the operator  $\hat{O}$ . Inserting (4.3) into the face factors of (3.9) we obtain what we call the *proper EPRL vertex amplitude*:

$$A_v^{(\Pi+)}(\{k_{ab}, \psi_{ab}\}) := \int_{\text{Spin}(4)^5} \prod_a dG_a \prod_{a < b} \epsilon(l_{k_{ab}}^{j_{ab}^-, j_{ab}^+} \psi_{ab}, \rho(G_{ab}) l_{k_{ab}}^{j_{ab}^-, j_{ab}^+} P_{ba}(\{G_{a'b'}\}) \psi_{ba}). \quad (4.4)$$

Let us stop for a moment and remark on the properties of this vertex amplitude. First, as the EPRL vertex, it depends on an  $SU(2)$  boundary state and hence may be used to construct a spin-foam model *for loop quantum gravity*. It is linear in the  $SU(2)$  boundary state, as required for the final spin-foam amplitude to be linear in the boundary state, or equivalently, sesquilinear in the initial and final states. Furthermore, as we will show in the next subsection, it is invariant under  $SU(2)$  gauge transformations. Finally, and most importantly, as we will show in the next section, its asymptotics only include the single factor  $e^{iS_{\text{Regge}}}$ , as desired.

Throughout the rest of this paper, the notation  $P_{ba}(\{G_{a'b'}\})$  introduced in (4.3) will also refer to the projector acting in the spin  $(s_{ab}^-, s_{ab}^+)$  representation of  $\text{Spin}(4)$ , defined by the same expression (4.3). In each statement using the notation  $P_{ba}(\{G_{a'b'}\})$ , either the context will determine which projector is intended, or the statement will hold for both projectors.

Finally, let us briefly note two ways to rewrite the proper vertex: (1.) It may at first appear arbitrary that the projector was inserted on the right side of each face factor in equation (4.4). However, in fact, one can put the projector (appropriately transformed) anywhere in each face-factor, and the vertex amplitude doesn't change. See appendix C. (2.) We note that, using equation (3.1), one has the following equivalent expression for the proper vertex:

$$A_v^{(\Pi+)}(\{k_{ab}, \psi_{ab}\}) := \int_{\text{Spin}(4)^5} \prod_a dG_a \prod_{a < b} \langle J l_{k_{ab}}^{j_{ab}^-, j_{ab}^+} \psi_{ab}, \rho(G_{ab}) l_{k_{ab}}^{j_{ab}^-, j_{ab}^+} P_{ba}(\{G_{a'b'}\}) \psi_{ba} \rangle. \quad (4.5)$$

## 4.2 Proof of invariance under $SU(2)$ gauge transformations

**Theorem 4.** *The proper EPRL vertex is invariant under arbitrary  $SU(2)$  gauge transformations at the tetrahedra.*

**Proof.** Let  $\{k_{ab}, \psi_{ab}\}$  be the data for a given spin network on the boundary, and let five  $SU(2)$  elements  $h_a$ , one at each tetrahedron, be given. We wish to show

$$A_v^{(\Pi+)}(\Psi_{\{k_{ab}, \rho(h_a) \psi_{ab}\}}) = A_v^{(\Pi+)}(\Psi_{\{k_{ab}, \psi_{ab}\}}).$$

First, define  $\tilde{G}_{ab} := (h_a, h_a)^{-1} \circ G_{ab} \circ (h_b, h_b)$ . Then

$$\begin{aligned} (\tilde{G}_{ab} \cdot \mathcal{N})^i &= \text{tr}(\tau^i \tilde{X}_{ab}^- \tilde{X}_{ba}^+) = \text{tr}(\tau^i h_a^{-1} X_{ab}^- X_{ba}^+ h_a) \\ &= \text{tr}((h_a \tau^i h_a^{-1}) X_{ab}^- X_{ba}^+) = h_a \triangleright \text{tr}(\tau^i X_{ab}^- X_{ba}^+) \\ &= h_a \triangleright (G_{ab} \cdot \mathcal{N})^i. \end{aligned} \quad (4.6)$$

From this and the  $SO(3)$  invariance of  $\epsilon_{ijk}$ , it follows that

$$\beta_{ab}(\{\tilde{G}_{a'b'}\}) = \beta_{ab}(\{G_{a'b'}\}). \quad (4.7)$$

We thus have

$$\begin{aligned} \rho(h_b)^{-1} P_{ba}(\{G_{a'b'}\}) \rho(h_b) &= \rho(h_b)^{-1} P_{(0, \infty)}(\beta_{ab}(\{G_{a'b'}\})(G_{ba} \cdot \mathcal{N})_i L^i) \rho(h_b) \\ &= P_{(0, \infty)}(\beta_{ab}(\{G_{a'b'}\})[(h_b)^{-1} \triangleright (G_{ba} \cdot \mathcal{N})_i] L^i) \\ &= P_{(0, \infty)}(\beta_{ab}(\{\tilde{G}_{a'b'}\})(\tilde{G}_{ba} \cdot \mathcal{N})_i L^i) \\ &= P_{ba}(\{\tilde{G}_{a'b'}\}) \end{aligned} \quad (4.8)$$

where lemma 8 has been used in the second line, and (4.6) and (4.7) have been used in the third. Using (4.8), we finally have

$$\begin{aligned}
A_v(\{k_{ab}, \rho(h_a)\psi_{ab}\}) &:= \int \left( \prod_{a<b} dG_{ab} \right) \prod_{a<b} \epsilon \left( \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} \rho(h_a)\psi_{ab}, \rho(G_{ab}) \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} P_{ba}(\{G_{a'b'}\}) \rho(h_b)\psi_{ba} \right) \\
&= \int \left( \prod_{a<b} dG_{ab} \right) \prod_{a<b} \epsilon \left( \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} \rho(h_a)\psi_{ab}, \rho(G_{ab}) \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} \rho(h_b) P_{ba}(\{\tilde{G}_{a'b'}\}) \psi_{ba} \right) \\
&= \int \left( \prod_{a<b} dG_{ab} \right) \prod_{a<b} \epsilon \left( \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} \psi_{ab}, \rho(h_a, h_a)^{-1} \rho(G_{ab}) \rho(h_b, h_b) \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} P_{ba}(\{\tilde{G}_{a'b'}\}) \psi_{ba} \right) \\
&= \int \left( \prod_{a<b} dG_{ab} \right) \prod_{a<b} \epsilon \left( \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} \psi_{ab}, \rho(\tilde{G}_{ab}) \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} P_{ba}(\{\tilde{G}_{a'b'}\}) \psi_{ba} \right) \\
&= \int \left( \prod_{a<b} d\tilde{G}_{ab} \right) \prod_{a<b} \epsilon \left( \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} \psi_{ab}, \rho(\tilde{G}_{ab}) \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} P_{ba}(\{\tilde{G}_{a'b'}\}) \psi_{ba} \right) \\
&= A_v(\{k_{ab}, \psi_{ab}\})
\end{aligned}$$

where we have used in the third line the intertwining property of  $\iota_{k_{ab}}^{s_{ab}^- s_{ab}^+}$  and in the second to last line the right and left invariance of the Haar measure.  $\blacksquare$

## 5 Asymptotics

In the following we state and prove the asymptotics of the proper vertex, using key results from [14].

### 5.1 Statement of the formula

It will be useful for later purposes to define the following before defining coherent states.

**Definition 2.** Given any unit  $n^i \in \mathbb{R}^3$ , let  $|n; k, m\rangle$  denote the eigenstate of  $n \cdot \hat{L}$  in  $V_k$  with eigenvalue  $m$ , and  $|n; j^-, j^+, k, m\rangle$  the eigenstate of  $\hat{L}^2$  and  $n \cdot \hat{L}$  in  $V_{j^-, j^+}$  with eigenvalues  $k(k+1)$  and  $m$ , with phase fixed arbitrarily for each set of labels.

**Definition 3.** Given a unit 3-vector  $n$ , a spin  $j$ , and a phase  $\theta$ , we define the corresponding coherent state as

$$|n, \theta\rangle_j := e^{i\theta} |n; j, j\rangle. \quad (5.1)$$

The  $\theta$  argument represents a phase freedom, and will usually be suppressed. Additionally, when the spin is clear from the context, it will be omitted.

We call an assignment of one spin  $k_{ab} \in \mathcal{K}_\gamma$  and two unit 3-vectors  $n_{ab}^i, n_{ba}^i$  to each triangle  $(ab)$  in  $S$  a set of *quantum boundary data*. Given such data, the corresponding boundary state in the  $SU(2)$  boundary Hilbert space of  $S$  is

$$\Psi_{\{k_{ab}, n_{ab}\}, \theta} := \Psi_{\{k_{ab}, \psi_{ab}\}} \quad \text{with} \quad |\psi_{ab}\rangle := |n_{ab}, \theta_{ab}\rangle_{k_{ab}} \quad (5.2)$$

where the  $\theta_{ab}$  are any phases summing to  $\theta$  modulo  $2\pi$ . The phase  $\theta$  will usually be suppressed. The state  $\Psi_{\{k_{ab}, n_{ab}\}}$  so defined is a coherent boundary state corresponding to the classical reduced boundary data  $A_{ab} = A(k_{ab}) := \kappa\gamma k_{ab}$  and  $n_{ab}$ .

Suppose  $\{A(k_{ab}), n_{ab}\}$  is non-degenerate and satisfies closure. Then we likewise say that  $\{k_{ab}, n_{ab}\}$  is non-degenerate and satisfies closure. In this case, for each tetrahedron  $a$ , there exists a geometrical tetrahedron in  $\mathbb{R}^3$ , unique up to translations, such that  $\{A(k_{ab})\}_{b \neq a}$  and  $\{n_{ab}^i\}_{b \neq a}$  are the areas and outward unit normals, respectively, of the four triangular faces, which we denote by  $\{\Delta_{ab}^t\}_{b \neq a}$ . If these five geometrical tetrahedra can be glued together consistently to form a 4-simplex, we say that the boundary data  $\{k_{ab}, n_{ab}\}$  is *Regge-like*. For such data, there exists a set of  $SU(2)$  elements  $\{g_{ab} = g_{ba}^{-1}\}$ , unique up to a  $\mathbb{Z}_2$  lift ambiguity [14], such that the adjoint action of each  $g_{ab}$  on  $\mathbb{R}^3$  maps (1.)  $\Delta_{ab}^t$  into  $\Delta_{ba}^t$ , and (2.)  $n_{ba}$  into  $-n_{ab}$ . These group elements can be used to completely remove the phase ambiguity in the boundary state (5.2), by requiring the phase of the coherent states to be chosen such that  $g_{ab}|n_{ba}\rangle_{k_{ab}} = J|n_{ab}\rangle_{k_{ab}}$ , where  $J$  is as defined in section 3.1. The resulting boundary state  $\Psi_{\{k_{ab}, n_{ab}\}}$  is called the *Regge state* determined by  $\{k_{ab}, n_{ab}\}$ , and is denoted by  $\Psi_{\{k_{ab}, n_{ab}\}}^{\text{Regge}}$ .

The following theorem, as theorem 1 in [14], uses the fact that, because the boundary data  $\{k_{ab}, n_{ab}\}$  determine the geometry of all boundary tetrahedra, it also determines the geometry of the 4-simplex itself [14, 24], and hence, in particular, the dihedral angles  $\Theta_{ab}$ . More precisely  $\Theta_{ab} \in [0, \pi]$  is defined by the equation  $N_a \cdot N_b = \cos \Theta_{ab}$  where  $N_a$  and  $N_b$  are the outward pointing normals to the  $a$ th and  $b$ th tetrahedra, respectively.

**Theorem 5** (Proper EPRL asymptotics). *If  $\{k_{ab}, n_{ab}\}$  is boundary data representing a non-degenerate Regge geometry, then*

$$A_v(\Psi_{\lambda k_{ab}, n_{ab}}^{\text{Regge}}) \sim \left(\frac{2\pi}{\lambda}\right)^{12} N_{+-}^\gamma \exp\left(i \sum_{a < b} A(\lambda k_{ab}) \Theta_{ab}\right) \quad (5.3)$$

where  $N_{+-}^\gamma$  is the Hessian factor calculated in [14]. If  $\{k_{ab}, n_{ab}\}$  does not represent a non-degenerate Regge geometry, then  $A_v(\Psi_{\lambda k_{ab}, n_{ab}, \theta})$  decays exponentially with large  $\lambda$  for any choice of phase  $\theta$ .

To prove this theorem, in manner similar to [14], we cast the proper vertex in appropriate integral form  $A_v = \int d\mu(x) e^{S_{\gamma < 1}(x)}$  and  $A_v = \int d\mu(x) e^{S_{\gamma > 1}(x)}$ , separately for the cases  $\gamma < 1$  and  $\gamma > 1$ , where  $S_{\gamma < 1}$  and  $S_{\gamma > 1}$  are “actions”. We then determine the critical points for each action. In proving this theorem, we are interested in critical points whose contributions are not exponentially suppressed. For this reason, we define the term “critical point” to mean points where the action is stationary and its real part is *non-negative*. If a point in the domain of integration is such that the real part is an absolute maximum and is non-negative, we shall say it is a *maximal point*.

## 5.2 Integral expressions and critical points

In the following, whenever we say the words “critical points” with no other qualification, we are referring to critical points of the proper EPRL vertex (4.4).

### 5.2.1 The case $\gamma < 1$

The relevant integral form of the proper vertex in this case is

$$A_v(\Psi_{k_{ab}, \psi_{ab}, \theta}) = \int \prod_a dX_a^- dX_a^+ \exp(S_{\gamma < 1}) \quad (5.4)$$

where

$$\exp(S_{\gamma < 1}) = \prod_{a < b} \langle J_{k_{ab}}^{s_{ab}^- s_{ab}^+} n_{ab}, \quad \rho(X_{ab}^-, X_{ab}^+) \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} P_{ba}(\{G_{a'b'}\}) n_{ba} \rangle. \quad (5.5)$$

The action  $S_{\gamma < 1}$  is, as in [14], generally complex. The two conditions that determine critical points are maximality and stationarity. In both proving the equations for maximality and checking stationarity,

it will be simplest to reuse the results in [14]. This will highlight the simplicity of the additional steps in reasoning necessary for the present modification. Recall from [14] that the action for  $\gamma < 1$  for the original EPRL model is

$$\exp(S_{\gamma < 1}^{\text{EPRL}}) = \prod_{a < b} \langle J \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} n_{ab}, \quad \rho(X_{ab}^-, X_{ab}^+)^{-1} \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} n_{ba} \rangle. \quad (5.6)$$

For the purpose of the following lemmas and the rest of this section, a set of group elements together with boundary data  $\{X_a^\pm, k_{ab}, n_{ab}\}$  is said to satisfy *proper orientation* if, for all  $a \neq b$ ,  $\beta_{ab}(\{G_{a'b'}\}) \text{tr}(\tau_i X_{ab}^- X_{ba}^+) n_{ab}^i > 0$ .

**Lemma 3.** *Given boundary data  $\{k_{ab}, n_{ab}\}$ ,  $\{X_a^\pm\}$  is a maximal point of  $S_{\gamma < 1}$  iff orientation and proper orientation are satisfied.*

**Proof.** Because (5.5) and (5.6) only differ by insertion of the projectors  $P_{ba}(\{G_{a'b'}\})$ , and recalling from [14] that  $|\exp(S_{\gamma < 1}^{\text{EPRL}})| \leq 1$ , one immediately has

$$\exp(2\text{Re } S_{\gamma < 1}) = |\exp(S_{\gamma < 1})| \leq |\exp(S_{\gamma < 1}^{\text{EPRL}})| \leq 1 \quad (5.7)$$

From [14], the second  $\leq$  is an equality iff orientation is satisfied. The first  $\leq$  is an equality iff the inserted projectors act as unity, i.e., iff  $P_{ba}(\{G_{a'b'}\})|n_{ba}\rangle = |n_{ba}\rangle$ , which, if orientation holds, is equivalent to proper orientation,  $\beta_{ab}(\{G_{a'b'}\}) \text{tr}(\tau_i X_{ab}^- X_{ba}^+) n_{ab}^i > 0$ . Thus,  $\{G_{a'b'}\}$  is a maximal point, so that both inequalities are saturated, iff orientation and proper orientation hold. ■

**Lemma 4.** *Let boundary data  $\{k_{ab}, n_{ab}\}$  be given, and suppose  $\{X_a^\pm\}$  is a maximal point of  $S_{\gamma < 1}$ . Then it is also a stationary point of  $S_{\gamma < 1}$  iff closure is additionally satisfied.*

**Proof.** If  $\delta$  is any variation of the group elements  $X_a^\pm$ , from (5.5), (5.6) and the fact that  $\{X_a^\pm\}$  is maximal, one immediately has

$$\delta \exp(S_{\gamma < 1}) = \delta \exp(S_{\gamma < 1}^{\text{EPRL}}) + \prod_{a < b} \langle J \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} n_{ab}, \quad \rho(X_{ab}^-, X_{ab}^+) \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} (\delta P_{ba}(\{G_{a'b'}\})) n_{ba} \rangle. \quad (5.8)$$

From lemma 7.c,

$$P_{ba}(\{G_{a'b'}\}) \circ \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} = \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} \circ P_{ba}(\{G_{a'b'}\}). \quad (5.9)$$

Taking the variation of both sides, and using the result with (5.8), we obtain

$$\delta \exp(S_{\gamma < 1}) = \delta \exp(S_{\gamma < 1}^{\text{EPRL}}) + \prod_{a < b} \langle J \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} n_{ab}, \quad \rho(X_{ab}^-, X_{ab}^+) (\delta P_{ba}(\{G_{a'b'}\})) \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} n_{ba} \rangle. \quad (5.10)$$

From the lemma 3, as  $\{X_a^\pm\}$  is a maximal point, orientation is satisfied. Using this,

$$\begin{aligned} \rho(X_{ba}^-, X_{ba}^+) J \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} |n_{ab}\rangle &\propto \rho(X_{ba}^-, X_{ba}^+) J |n_{ab}; s_{ab}^-, s_{ab}^+, k_{ab}, k_{ab}\rangle \\ &\propto \rho(X_{ba}^-, X_{ba}^+) | - n_{ab}; s_{ab}^-, s_{ab}^+, k_{ab}, k_{ab}\rangle \\ &\propto \rho(X_{ba}^-, X_{ba}^+) [| - n_{ab}; s_{ab}^-, s_{ab}^- \rangle \otimes | - n_{ab}; s_{ab}^+, s_{ab}^+ \rangle] \\ &\propto | - X_{ba}^- \triangleright n_{ab}; s_{ab}^-, s_{ab}^- \rangle \otimes | - X_{ba}^+ \triangleright n_{ab}; s_{ab}^+, s_{ab}^+ \rangle \\ &\propto |n_{ba}; s_{ab}^-, s_{ab}^- \rangle \otimes |n_{ba}; s_{ab}^+, s_{ab}^+ \rangle \propto |n_{ba}; s_{ab}^-, s_{ab}^+, k_{ab}, k_{ab}\rangle \end{aligned} \quad (5.11)$$

Where lemma 7.b was used in line 1 and  $k_{ab} = s_{ab}^- + s_{ab}^+$ , was used in lines 3 and 5.

We next claim that  $|n_{ba}; s_{ab}^-, s_{ab}^+, k_{ab}, k_{ab}\rangle$  is an eigenstate of  $\text{tr}(\tau_i X_{ab}^- X_{ba}^+) \hat{L}^i$ . If  $X_{ab}^- X_{ba}^+ = \pm I$ , then  $\text{tr}(\tau_i X_{ab}^- X_{ba}^+) = 0$  and the claim is trivially true. If, on the other hand,  $X_{ab}^- X_{ba}^+ \neq \pm I$ , then from equation (52) in [14],  $X_{ab}^- X_{ba}^+ = \exp(\lambda_{ab} n_{ab} \cdot \tau)$  for some  $\lambda_{ab}$ , so that  $n_{ab}^i = \pm n [X_{ab}^- X_{ba}^+]^i \hat{= \text{tr}(\tau_i X_{ab}^- X_{ba}^+)}$ , and the claim follows from the definition of  $|n_{ba}; s_{ab}^-, s_{ab}^+, k_{ab}, k_{ab}\rangle$ . In either case, the claim is proven.

This, along with (5.11),  $\iota_{k_{ab}}^{s_{ab}^-, s_{ab}^+} |n_{ab}\rangle = |n_{ba}; s_{ab}^-, s_{ab}^+, k_{ab}, k_{ab}\rangle$ ,  $P_{ba} = P_{(0,\infty)}(\beta_{ab} n [X_{ba}^- X_{ab}^+] \cdot L)$  and corollary 9 in appendix B, implies that the second term in (5.10) is zero. As proven in [14], using that orientation is satisfied, the remaining term in (5.10) is zero iff closure is satisfied. ■

**Theorem 6.** *Given boundary data  $\{k_{ab}, n_{ab}\}$ ,  $\{X_a^\pm\}$  is a critical point of  $S_{\gamma < 1}$  iff closure, orientation, and proper orientation are satisfied.*

**Proof.**

( $\Rightarrow$ ) Suppose  $\{X_a^\pm\}$  is a critical point of  $S_{\gamma < 1}$ . Then lemma 3 implies that orientation and proper orientation are satisfied, and lemma 4 implies that closure is satisfied.

( $\Leftarrow$ ) Suppose closure, orientation, and proper orientation are satisfied. Then by lemma 3,  $\{X_a^\pm\}$  is a maximal point of  $S_{\gamma < 1}$ , and by lemma 4 it is a stationary point of  $S_{\gamma < 1}$ . ■

### 5.2.2 The case $\gamma > 1$

For this case, we derive from scratch an expression for the proper vertex analogous to (18) and (19) in [14]. In doing this, we use the spinorial form of the irreps of  $SU(2)$ . Let  $A, B, C, \dots = 0, 1$  denote spinor indices. The carrying space  $V_j$  can then be realized as the space of symmetric spinors of rank  $2j$  (see, for example, [4]). Let  $n^A$  denote the spinor corresponding to the coherent state  $|n\rangle_{\frac{1}{2}}$ . As in [14, 25], the key property of coherent states we use is that, in their spinorial form, the higher spin coherent states are given by

$$(|n\rangle_j)^{A_1 \dots A_{2j}} = n^{A_1} \dots n^{A_{2j}}. \quad (5.12)$$

From the relation (3.8) between  $k$  and  $s^+$ ,  $s^-$  for a given triangle, one deduces for  $\gamma > 1$  that  $s^+ = s^- + k$ . For this case, the explicit expression for  $\iota_k^{s^-, s^+}$  in terms of symmetric spinors is given in equations (A.12) and (A.13) of [4]<sup>3</sup>. Let  $v^{A_1 \dots A_{2k}} \in V_k$  be given. For  $\gamma > 1$ , one has

$$\iota_k^{s^-, s^+}(v)^{A_1 \dots A_{2s^+} B_1 \dots B_{2s^-}} = v^{(A_1 \dots A_{2k}} \epsilon^{A_{2k+1} | B_1 |} \dots \epsilon^{A_{2s^+} | B_{2s^-}} \quad (5.13)$$

where the symmetrization is over the  $A$  indices only. In order to impose the symmetrization over the  $A$  indices, similar to [14], on the left of each  $\iota_k^{s^-, s^+}$ , acting in the self-dual part of the co-domain, we insert a resolution of the identity on  $V_{s^+}$  into coherent states:

$$d_{s^+} \int dm |m\rangle_{s^+} \langle m| = I_{s^+} \quad (5.14)$$

where  $dm$  is the measure on the metric 2-sphere normalized to unit area, and  $d_s := 2s + 1$ . In spinorial notation

$$d_{s^+} \int dm m^{A_1} \dots m^{A_{2s^+}} m_{B_1}^\dagger \dots m_{B_{2s^+}}^\dagger = \delta_{B_1}^{(A_1} \dots \delta_{B_{2s^+}}^{A_{2s^+})}. \quad (5.15)$$

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<sup>3</sup>In (A.13) of [4], symmetrization over the  $A$  group,  $B$  group, and  $C$  group of indices was forgotten but was clear from the context.

where  $m_A^\dagger := (\frac{1}{2}\langle m|)_A$ . Starting from equation (4.4) with  $\psi_{ab} = |n_{ab}\rangle_{k_{ab}} = n_{ab}^{A_1} \dots n_{ab}^{A_{2k_{ab}}}$ , writing out all spinor indices explicitly, we insert two resolutions of the identity (5.15) into each face factor in (4.4), one after each  $\iota_k^{s^- s^+}$ . Denote the integration variables  $m_{ab}$  and  $m_{ba}$  respectively for the left and right insertions. Writing out the  $\epsilon$ -inner product in terms of alternating tensors  $\epsilon_{AB}$ , using  $m_A^\dagger = \epsilon_{AB}(Jm)^B$ , simplifying, and then writing the final expression again in terms of hermitian inner products, one obtains

$$A_v = \int \prod_a dX_a^+ dX_a^- \left( \prod_{ab} dm_{ab} d_{s_{ab}^+}^2 \right) \exp(S_{\gamma>1}) \quad (5.16)$$

where

$$\begin{aligned} \exp(S_{\gamma>1}) &= \prod_{a<b} k_{ab} \langle m_{ab} | n_{ab} \rangle_{k_{ab} s_{ab}^+} \langle Jm_{ab} | \rho(X_{ab}^+) | m_{ba} \rangle_{s_{ab}^+} \\ &\quad \overline{k_{ab} \langle m_{ba} | P_{ba}(\{G_{a'b'}\}) | n_{ba} \rangle_{k_{ab} s_{ab}^-} \langle Jm_{ab} | \rho(X_{ab}^-) | m_{ba} \rangle_{s_{ab}^-}}. \end{aligned} \quad (5.17)$$

Recall from [14] that the action for  $\gamma > 1$  for the original EPRL model is<sup>4</sup>

$$\begin{aligned} \exp(S_{\gamma>1}^{\text{EPRL}}) &= \prod_{a<b} k_{ab} \langle m_{ab} | n_{ab} \rangle_{k_{ab} s_{ab}^+} \langle Jm_{ab} | \rho(X_{ab}^+) | m_{ba} \rangle_{s_{ab}^+} \\ &\quad \overline{k_{ab} \langle m_{ba} | n_{ba} \rangle_{k_{ab} s_{ab}^-} \langle Jm_{ab} | \rho(X_{ab}^-) | m_{ba} \rangle_{s_{ab}^-}}. \end{aligned} \quad (5.18)$$

**Lemma 5.** *Given boundary data  $\{k_{ab}, n_{ab}\}$ ,  $\{X_a^\pm, m_{ab}\}$  is a maximal point of  $S_{\gamma>1}$  iff orientation and proper orientation are satisfied and  $m_{ab} = n_{ab}$  for all  $a \neq b$ .*

**Proof.** Because (5.17) and (5.18) only differ by insertion of the projectors  $P_{ba}$ , and recalling from [14] that  $|\exp(S_{\gamma>1}^{\text{EPRL}})| \leq 1$ , one has

$$\exp(2\text{Re } S_{\gamma>1}) = |\exp(S_{\gamma>1})| \leq |\exp(S_{\gamma>1}^{\text{EPRL}})| \leq 1. \quad (5.19)$$

From [14], the second  $\leq$  is an equality iff orientation is satisfied and  $m_{ab} = n_{ab}$  for all  $a \neq b$ . As in the  $\gamma < 1$  case, the first  $\leq$  is an equality iff the inserted projectors act as unity, which, if orientation is satisfied, is equivalent to proper orientation. It follows that  $\{G_{a'b'}\}$  is a maximal point, so that both inequalities are saturated, iff orientation, proper orientation, and  $m_{ab} = n_{ab}$  for all  $a \neq b$ , are satisfied. ■

**Lemma 6.** *Let boundary data  $\{k_{ab}, n_{ab}\}$  be given, and suppose  $\{X_a^\pm, m_{ab}\}$  is a maximal point of  $S_{\gamma>1}$ . Then it is also a stationary point of  $S_{\gamma>1}$  iff closure is additionally satisfied.*

**Proof.** If  $\delta$  is any variation of  $X_a^\pm$  and  $m_{ab}$ , from (5.17) and (5.18) one has

$$\begin{aligned} \delta \exp(S_{\gamma>1}) &= \delta \exp(S_{\gamma>1}^{\text{EPRL}}) + \prod_{a<b} k_{ab} \langle m_{ab} | n_{ab} \rangle_{k_{ab} s_{ab}^+} \langle Jm_{ab} | \rho(X_{ab}^+) | m_{ba} \rangle_{s_{ab}^+} \\ &\quad \overline{k_{ab} \langle m_{ba} | (\delta P_{ba}(\{G_{a'b'}\})) | n_{ba} \rangle_{k_{ab} s_{ab}^-} \langle Jm_{ab} | \rho(X_{ab}^-) | m_{ba} \rangle_{s_{ab}^-}} \end{aligned} \quad (5.20)$$

Because  $\{X_a^\pm, m_{ab}\}$  is a maximal point, from lemma 5, orientation and proper orientation are satisfied, and  $m_{ab} = n_{ab}$  for all  $a \neq b$ . It follows that  $|n_{ba}\rangle_{k_{ab}} = |m_{ba}\rangle_{k_{ab}}$  and both are eigenstates of

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<sup>4</sup>The coherent state  $|m_{ab}\rangle$  used here is related to the corresponding coherent state used in [14] by the action of  $J$ .

$P_{ba}(\{G_{a'b'}\})$  with eigenvalue 1, so that by corollary 9 in appendix C, the second term above is zero. As proven in [14], because orientation is satisfied and  $m_{ab} = n_{ab}$  for all  $a \neq b$ , it follows that the remaining term in (5.20) is zero iff closure is satisfied. ■

**Theorem 7.** *Given boundary data  $\{k_{ab}, n_{ab}\}$ ,  $\{X_a^\pm, m_{ab}\}$  is a critical point of  $S_{\gamma>1}$  iff closure, orientation, and proper orientation are satisfied, and  $m_{ab} = n_{ab}$  for all  $a \neq b$ .*

**Proof.**

( $\Rightarrow$ ) Suppose  $\{X_a^\pm, m_{ab}\}$  is a critical point of  $S_{\gamma>1}$ . Then lemma 5 implies that orientation and proper orientation are satisfied and  $m_{ab} = n_{ab}$  for all  $a \neq b$ , and lemma 6 implies that closure is satisfied.

( $\Leftarrow$ ) Suppose closure, orientation, and proper orientation are satisfied and  $m_{ab} = n_{ab}$  for all  $a \neq b$ . Then by lemma 5,  $\{X_a^\pm, m_{ab}\}$  is a maximal point of  $S_{\gamma>1}$ , and by lemma 6 it is a stationary point of  $S_{\gamma>1}$ . ■

Thus, though, in the  $\gamma > 1$  case, one has an extra set of variables  $\{m_{ab}\}$ , these are restricted to be equal to  $\{n_{ab}\}$  by the critical point equations, allowing one to treat the  $\gamma < 1$  and  $\gamma > 1$  cases in a unified way. The remaining critical point conditions on  $\{X_a^\pm, n_{ab}\}$  (given in theorem 6) have a symmetry: if  $\{X_a^\pm\}$  form a solution, then so does the set of group elements

$$\tilde{X}_a^\pm = \epsilon_a^\pm Y^\pm X_a^\pm \quad (5.21)$$

for any  $(Y^-, Y^+) \in Spin(4)$  and any set of ten signs  $\epsilon_a^\pm$ . This transformation is also a symmetry of the actions (5.5) and (5.17). If two solutions  $\{X_a^\pm\}, \{\tilde{X}_a^\pm\}$  are related by such a symmetry transformation, we call them *equivalent* and write  $\{X_a^\pm\} \sim \{\tilde{X}_a^\pm\}$ .

### 5.3 Proof of the asymptotic formula

Using the above results, we proceed to prove theorem 5.

Before getting into the details of the proof, we summarize its general structure. As already mentioned, the critical point equations for the proper vertex integrals (5.4) and (5.16) have a set of symmetries (5.21), of which the global  $Spin(4)$  symmetry is the only continuous one. In order to apply the stationary phase method to calculate the asymptotics, the critical points must be isolated, and hence this continuous symmetry must be removed. As in [14], we do this by performing the change of variables  $\tilde{G}_a := (G_0)^{-1} G_a$  for  $a = 1, \dots, 4$ . Then  $G_0$  no longer appears in the integrand, so that the  $G_0$  integral drops out. Upon removing the tilde labels, the remaining integrand is the same as the original integrand except with  $G_0$  replaced by the identity. It what follows  $G_a = (X_a^-, X_a^+)$  shall denote these “gauge-fixed” group elements, with  $G_0 \equiv \text{id}$ , in terms of which the continuous symmetry has been removed.

The proof then has two steps, the first of which has already been done in theorems 6 and 7 above: (1.) prove that the critical points of proper EPRL are precisely the subset of critical points of original EPRL at which proper orientation is satisfied. (2.) prove that, given a set of  $SU(2)$  boundary data  $\{k_{ab}, n_{ab}\}$ , the critical points of original EPRL at which proper orientation is satisfied are all equivalent and are precisely the critical points which give rise to the asymptotic term (5.3) in the original EPRL asymptotics [14]. Because proper orientation is satisfied, the value of the proper EPRL action at these critical points will be the same as the value of the original EPRL action at these points, yielding precisely the asymptotic behavior (5.3) claimed.

Let us begin by reviewing the results from theorems 6 and 7. The critical point equations for  $\gamma < 1$  and  $\gamma > 1$  are equivalent: the only difference is that for  $\gamma > 1$  one integrates over extra variables,  $m_{ab}$ ,



which, however, come with the critical point equations  $m_{ab} = n_{ab}$ , eliminating them. This allows us to effectively consider both the  $\gamma < 1$  case and  $\gamma > 1$  case simultaneously in the following. As given in theorems 6 and 7, the remaining critical point equations are

$$X_a^\pm \triangleright n_{ab} = -X_b^\pm \triangleright n_{ba} \quad (5.22)$$

and

$$\beta_{ab} \text{tr}(\tau_i X_{ab}^- X_{ba}^+) L_{ab}^i > 0 \quad (5.23)$$

for all  $a < b$ . The first of these, (5.22), is of the same form for both  $\{X_a^+\}$  and  $\{X_a^-\}$ :

$$U_a \triangleright n_{ab} = -U_b \triangleright n_{ba}. \quad (5.24)$$

One therefore proceeds by finding the solutions  $\{U_a\}$  to (5.24) for a given set of  $SU(2)$  boundary data  $\{k_{ab}, n_{ab}\}$ , and then from these one constructs the solutions  $\{X_a^\pm\}$  to (5.22), and then one checks which among these, if any, solves (5.23) in order to determine the critical points of the vertex integral.

The solutions to (5.24) have already been analyzed by [14]. To use the results of this analysis, one needs the notion of a *vector geometry*: A set of boundary data  $\{k_{ab}, n_{ab}\}$  is called a *vector geometry* if it satisfies closure and there exists  $\{h_a\} \subset SO(3)$  such that  $(h_a \triangleright n_{ab})^i = -(h_b \triangleright n_{ba})^i$  for all  $a \neq b$ . [14] then proceed by considering separately the three cases in which the boundary data (i.) does not define a vector geometry (ii.) defines a vector geometry which is, however, not a non-degenerate 4-simplex geometry, and (iii.) defines a non-degenerate 4-simplex geometry. We use this same division and consider each of these three cases in turn.

*Case (i.): Not a vector geometry.*

In this case, as proven in [14], there are no solutions to (5.24) and hence no solutions to (5.22), and hence no critical points. The vertex integral therefore decays exponentially with  $\lambda$ .

*Case (ii.): A vector geometry, but no non-degenerate 4-simplex geometry.*

In this case, as proven in [14], there is exactly one solution to (5.24), upto the equivalence (2.13). The only solution to (5.22) is therefore  $(X_a^-, X_a^+) = (U_a, \epsilon_a Y U_a)$ . But then  $X_{ba}^- X_{ab}^+ = \pm I$ , so that this solution fails to satisfy condition (5.23), so that there are no critical points. The vertex integral therefore decays exponentially with  $\lambda$ .

*Case (iii.): A non-degenerate 4-simplex geometry.*

In this case, as proven in [14], (5.24) has two inequivalent solutions  $\{U_a^1\}$  and  $\{U_a^2\}$ , so that there are four inequivalent solutions to (5.22):  $(X_a^-, X_a^+) = (U_a^1, U_a^1), (U_a^2, U_a^2), (U_a^1, U_a^2), (U_a^2, U_a^1)$ . Neither  $(U_a^1, U_a^1)$  nor  $(U_a^2, U_a^2)$ , nor any solution equivalent to these, satisfies (5.23), again because  $X_{ba}^- X_{ab}^+ = \pm I$ . It remains only to consider the solutions

$$\begin{aligned} (\overset{1}{X}_a^-, \overset{1}{X}_a^+) &= (U_a^1, U_a^2) \\ (\overset{2}{X}_a^-, \overset{2}{X}_a^+) &= (U_a^2, U_a^1). \end{aligned} \quad (5.25)$$

Because  $\overset{1}{X}_{ab}^- \overset{1}{X}_{ba}^+ = \left( \overset{2}{X}_{ab}^- \overset{2}{X}_{ba}^+ \right)^{-1}$ , the proper axes  $n[\overset{1}{X}_{ab}^- \overset{1}{X}_{ba}^+]^i, n[\overset{2}{X}_{ab}^- \overset{2}{X}_{ba}^+]^i$  defined in (2.22) are equal and opposite, so that

$$\text{tr}(\tau_i \overset{1}{X}_{ab}^- \overset{1}{X}_{ba}^+) = -\text{tr}(\tau_i \overset{2}{X}_{ab}^- \overset{2}{X}_{ba}^+). \quad (5.26)$$

From this one deduces

$$\beta_{ab}(\{\overset{1}{G}_{a'b'}\}) = \beta_{ab}(\{\overset{2}{G}_{a'b'}\}) \quad (5.27)$$

which gives us

$$\beta_{ab}(\{\overset{1}{G}_{a'b'}\}) \text{tr}(\tau_i \overset{1}{X}_{ab}^- \overset{1}{X}_{ba}^+) L_{ba}^i = -\beta_{ab}(\{\overset{2}{G}_{a'b'}\}) \text{tr}(\tau_i \overset{2}{X}_{ab}^- \overset{2}{X}_{ba}^+) L_{ba}^i. \quad (5.28)$$

Because  $\{U_a^1\} \not\sim \{U_a^2\}$ , we have  $\{\dot{X}_a^+\} \not\sim \{\dot{X}_a^-\}$  and  $\{\ddot{X}_a^+\} \not\sim \{\ddot{X}_a^-\}$ , so that both  $\{\dot{X}_a^\pm\}$  and  $\{\ddot{X}_a^\pm\}$  satisfy the hypotheses of lemma 2, implying that neither side of (5.28) is zero. It follows that exactly one of  $\beta_{ab}(\{\dot{G}_{a'b'}\})\text{tr}(\tau_i \ddot{X}_{ab}^- \ddot{X}_{ba}^+) L_{ba}^i$ ,  $\alpha = 1, 2$ , is positive, so that exactly one of  $\{\dot{X}_a^\pm\}$ ,  $\{\ddot{X}_a^\pm\}$  satisfies proper orientation and so is a critical point. Furthermore, at this one critical point,  $\mu = 1$ , so that, because the value of the action (5.5) (respectively (5.17)) for the proper vertex is equal to the value of the action (5.6) (respectively (5.18)) for the original vertex, from the analysis of [14], this one critical point gives rise to precisely the desired asymptotics stated in theorem 5.

## 6 Conclusions

The original EPRL model, as pointed out in [16], due to the fact that it is based on the linear simplicity constraints, necessarily mixes three of what we call Plebanski sectors<sup>5</sup>, only one of which — what has been called the (II+) sector — yields gravity with the usual action. This mixing of Plebanski sectors was identified as the precise reason for the presence of undesired terms in the asymptotics of the EPRL vertex calculated in [14]. Furthermore, as noted in section 3.2, this fact that Plebanski sectors are mixed impedes an unambiguous identification of the  $SU(2)$  boundary operators with those of LQG.

In this paper, a solution to this problem is found. We began by deriving a classical discrete condition that isolates the one desired gravitational sector (II+). By appropriately quantizing this condition and using it to modify the EPRL vertex amplitude, we have constructed what we call the *proper* EPRL vertex amplitude. This vertex amplitude continues to be a function of  $SU(2)$  spin-network data, so that it may continue to be used to define dynamics for LQG. We have shown that the proper vertex is  $SU(2)$  gauge invariant and is linear in the boundary state, as required to ensure that the final transition amplitude is linear in the initial state and antilinear in the final state. Finally, it has the correct asymptotics with the *single* required term consisting in the exponential of  $i$  times the Regge action.

Two interesting extensions of this work would be (1.) to the Lorentzian signature and (2.) to an amplitude for an arbitrary 4-cell, which might be used in a spin-foam model involving arbitrary cell-complexes, similar to the generalization [12] of Kamiński, Kisielowski, and Lewandowski. The first of these extensions we expect to be straightforward. The second, however, seems to require a new way of thinking about the discrete constraint (2.25) used to isolate the (II+) sector. For, the  $\beta_{ab}$  sign factor involved in this condition uses in a central way the fact that there are 5 tetrahedra in each 4-simplex.

In closing, let us remark on a more common viewpoint on the problem of the presence of extra terms in the asymptotics of EPRL and other spin-foam models: that they should be eliminated by choosing the boundary state to be appropriately peaked on *both fluxes and connections* [26–28], for example by using *holomorphic coherent states* [26, 29, 30]. Although choosing such a boundary state does succeed in isolating a single term in the asymptotics for the case of a single 4-simplex, for more general triangulations it is not yet clear. Recent work [31, 32] has investigated the asymptotics of transition amplitudes for the EPRL model for general triangulations with boundary, but, thus far, only coherent states peaked on the fluxes alone have been used, so that, as expected, one still has contributions from different Plebanski sectors in the asymptotics. Whether or not the use of holomorphic coherent states on the boundary of general triangulations will be sufficient to isolate a single exponential term in the asymptotics of the transition amplitude thus remains to be seen. If the prescription does work, it would be very interesting to understand if and how it might be related

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<sup>5</sup>At the level of discrete analogies, an awareness of this mixing of three sectors was manifest already as early as [15].

to the proper vertex amplitude proposed here. If it does not work, then the proper vertex amplitude proposed here provides a valuable candidate for the vertex amplitude of quantum gravity with all of the strengths of the EPRL vertex mentioned above, but without the presence of extra sectors in the asymptotics.

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## A Closure relations

The following property is mentioned, for example, in [15, 33, 34].

**Theorem 8.** *For any geometrical 4-simplex  $\sigma$  in  $\mathbb{R}^4$ ,*

$$\sum_t V_t N_t^I = 0 \quad (\text{A.1})$$

where the sum is over tetrahedra, and  $V_t$  and  $N_t^I$  are the volume and outward normal to tetrahedron  $t$ .

**Proof.** Define  ${}^3\epsilon^I$  to be the three-form on  $\mathbb{R}^4$  with components  $({}^3\epsilon^I)_{JKL} = \epsilon^I_{JKL}$ . Then

$$d{}^3\epsilon^I = 0.$$

Thus,

$$0 = \int_{\sigma} d{}^3\epsilon^I = \sum_t \int_t {}^3\epsilon^I. \quad (\text{A.2})$$

Let  ${}^t\epsilon$  denote the volume form for  $t$ , so that for each  $t$ ,

$$\epsilon_{IJKL} = 4(N_t)_I ({}^t\epsilon)_{JKL}.$$

Pulling back  $JKL$  to tetrahedron  $t$ , it follows that

$${}^3\epsilon^I_{t \leftarrow} = N_t^I ({}^t\epsilon)$$

which combined with A.2 yields the result. ■

## B Properties of embeddings and projectors

Recall  $|n; k, m\rangle$  denotes the eigenstate of  $n \cdot \hat{L}$  in  $V_k$  with eigenvalue  $m$ , and  $|n; j^-, j^+, k, m\rangle$  the eigenstate of  $\hat{L}^2$  and  $n \cdot \hat{L}$  in  $V_{j^-, j^+}$  with eigenvalues  $k(k+1)$  and  $m$ ,

**Lemma 7.**

(a.)

$$\hat{L}^i \circ \iota_k^{j^-, j^+} = \iota_k^{j^-, j^+} \circ \hat{L}^i \quad (\text{B.1})$$

(b.) For each unit  $n^i \in \mathbb{R}^3$  and each  $k, m$ , there exists  $\theta \in \frac{\mathbb{R}}{2\pi\mathbb{Z}}$  such that

$$\iota_k^{j^-, j^+} |n; k, m\rangle = e^{i\theta} |n; j^-, j^+, k, m\rangle. \quad (\text{B.2})$$

(c.) For any  $\mathcal{S} \subseteq \mathbb{R}$ ,

$$P_{\mathcal{S}}(n \cdot \hat{L}) \circ \iota_k^{j^-, j^+} = \iota_k^{j^-, j^+} \circ P_{\mathcal{S}}(n \cdot \hat{L}). \quad (\text{B.3})$$

**Proof.**

*Proof of (a.):*

From the intertwining property of  $\iota_k^{j^-, j^+}$ ,

$$\rho(e^{t\tau^i}, e^{t\tau^i}) \iota_k^{j^-, j^+} = \iota_k^{j^-, j^+} \rho(e^{t\tau^i}). \quad (\text{B.4})$$

Taking  $i \frac{d}{dt}$  of both sides and setting  $t = 0$  yields the result.

*Proof of (b.):*

Using part (a.),

$$\begin{aligned} (n \cdot \hat{L}) \iota_k^{j^-, j^+} |n; k, m\rangle &= \iota_k^{j^-, j^+} (n \cdot \hat{L}) |n; k, m\rangle = m \iota_k^{j^-, j^+} |n; k, m\rangle, \quad \text{and} \\ \hat{L}^2 \iota_k^{j^-, j^+} |n; k, m\rangle &= \iota_k^{j^-, j^+} \hat{L}^2 |n; k, m\rangle = k(k+1) \iota_k^{j^-, j^+} |n; k, m\rangle. \end{aligned}$$

The result follows.

*Proof of (c.):*

We have for each  $m$ ,

$$\begin{aligned} P_{\mathcal{S}}(n \cdot \hat{L}) \iota_k^{j^-, j^+} |n; k, m\rangle &= e^{i\theta} P_{\mathcal{S}}(n \cdot \hat{L}) |n; j^-, j^+, k, m\rangle = e^{i\theta} \chi_{\mathcal{S}}(m) |n; j^-, j^+, k, m\rangle \\ &= \chi_{\mathcal{S}}(m) \iota_k^{j^-, j^+} |n; k, m\rangle = \iota_k^{j^-, j^+} P_{\mathcal{S}}(n \cdot \hat{L}) |n; k, m\rangle \end{aligned}$$

where  $\chi_{\mathcal{S}}(m)$  denotes the characteristic function for  $\mathcal{S}$ . ■

**Lemma 8.** In any irreducible representation (irrep) of  $Spin(4)$ , for any two  $(v_-)^i, (v_+)^i \in \mathbb{R}^3$

$$\rho(X^-, X^+) \circ P_{\mathcal{S}}(v_- \cdot \hat{J}^- + v_+ \cdot \hat{J}^+) = P_{\mathcal{S}}\left((X^- \triangleright v_-) \cdot \hat{J}^- + (X^+ \triangleright v_+) \cdot \hat{J}^+\right) \rho(X^-, X^+) \quad (\text{B.5})$$

**Proof.** Let  $j^{\pm}, m^{\pm}$  be given. Write  $v_{\pm}^i = \lambda_{\pm} n_{\pm}^i$  with  $\lambda_{\pm} \geq 0$  and  $n_{\pm}^i$  unit. Using that

$\rho(X^{\pm}) |n_{\pm}; j^{\pm}, m^{\pm}\rangle = e^{i\theta^{\pm}} |X^{\pm} \triangleright n_{\pm}; j^{\pm}, m^{\pm}\rangle$  for some  $\theta^{\pm}$ , we have

$$\begin{aligned} &\rho(X^-, X^+) P_{\mathcal{S}}(v_- \cdot \hat{J}^- + v_+ \cdot \hat{J}^+) |n_-; j^-, m^-\rangle \otimes |n_+; j^+, m^+\rangle \\ &= \chi_{\mathcal{S}}(\lambda_- m^- + \lambda_+ m^+) \rho(X^-, X^+) |n_-; j^-, m^-\rangle \otimes |n_+; j^+, m^+\rangle \\ &= e^{i(\theta^- + \theta^+)} \chi_{\mathcal{S}}(\lambda_- m^- + \lambda_+ m^+) |X^- \triangleright n_-; j^-, m^-\rangle \otimes |X^+ \triangleright n_+; j^+, m^+\rangle \\ &= e^{i(\theta^- + \theta^+)} P_{\mathcal{S}}\left((X^- \triangleright v_-) \cdot \hat{J}^- + (X^+ \triangleright v_+) \cdot \hat{J}^+\right) |X^- \triangleright n_-; j^-, m^-\rangle \otimes |X^+ \triangleright n_+; j^+, m^+\rangle \\ &= P_{\mathcal{S}}\left((X^- \triangleright v_-) \cdot \hat{J}^- + (X^+ \triangleright v_+) \cdot \hat{J}^+\right) \rho(X^-, X^+) |n_-; j^-, m^-\rangle \otimes |n_+; j^+, m^+\rangle \end{aligned}$$
■

**Lemma 9.** Let  $\hat{O}_t$  be any one-parameter family of self-adjoint operators on a Hilbert space  $\mathcal{H}$ . For each  $t$ , let  $\psi_t$  be a normalized eigenstate of  $\hat{O}_t$  such that all  $\psi_t$  have the same eigenvalue  $\lambda \in \mathbb{R}$ . Then

$$\langle \psi_t | \left( \frac{d}{dt} \hat{O}_t \right) | \psi_t \rangle = 0. \quad (\text{B.6})$$

**Proof.**

$$\langle \psi_t | \hat{O}_t | \psi_t \rangle = \lambda \quad (\text{B.7})$$

for all  $t$ . Taking  $\frac{d}{dt}$  of both sides,

$$\begin{aligned} \left( \frac{d}{dt} \langle \psi_t | \right) \hat{O}_t | \psi_t \rangle + \langle \psi_t | \left( \frac{d}{dt} \hat{O}_t \right) | \psi_t \rangle + \langle \psi_t | \hat{O}_t \frac{d}{dt} | \psi_t \rangle &= 0 \\ \lambda \frac{d}{dt} (\langle \psi_t, \psi_t \rangle) + \langle \psi_t | \left( \frac{d}{dt} \hat{O}_t \right) | \psi_t \rangle &= 0 \\ \langle \psi_t | \left( \frac{d}{dt} \hat{O}_t \right) | \psi_t \rangle &= 0 \end{aligned}$$

■

Applying this to the family of operators  $\hat{O}_t = n_t \cdot \hat{L}$  on  $V_{j^-, j^+}$  and the states  $|n_t; j^-, j^+, k, m\rangle$ , and to the family of operators  $\hat{O}_t = n_t \cdot \hat{L}$  on  $V_k$  and the states  $|n_t; k, m\rangle$ , yields the following.

**Corollary 9.** For any variation  $\delta$  of  $n$ , any  $j^-, j^+, k$ , any  $m \in \{-k, -k+1, \dots, k\}$ , and any set  $S \subset \mathbb{R}$ , one has

$$\langle n; j^-, j^+, k, m | \delta P_S(n \cdot \hat{L}) | n; j^-, j^+, k, m \rangle = 0. \quad (\text{B.8})$$

and

$$\langle n; k, m | \delta P_S(n \cdot \hat{L}) | n; k, m \rangle = 0. \quad (\text{B.9})$$

## C Expression for vertex with projectors on the left

**Lemma 10.** For each unit  $n^i \in \mathbb{R}^3$ ,  $g \in SU(2)$ ,  $k$ , and  $m$ , there exists  $\theta \in \frac{\mathbb{R}}{2\pi\mathbb{Z}}$  such that

$$\rho(h) |n; k, m\rangle = e^{i\theta} |h \triangleright n; k, m\rangle \quad (\text{C.1})$$

**Proof.**

$$\left( (h \triangleright n) \cdot \hat{L} \right) \rho(h) |n; k, m\rangle = \rho(h) (n \cdot \hat{L}) |n; k, m\rangle = m \rho(h) |n; k, m\rangle.$$

■

**Lemma 11.** For any  $S \subseteq \mathbb{R}$ , and in any irrep of  $Spin(4)$ , and any  $v^i \in \mathbb{R}^3$ ,

$$P_S(v \cdot \hat{L}) \circ J = J \circ P_S(-v \cdot \hat{L}) \quad (\text{C.2})$$

**Proof.** Let  $v^i =: \lambda n^i$  with  $\lambda \geq 0$  and  $n^i$  unit. Using that  $J$  anti-commutes with  $\hat{L}^i$ , for any  $n$  and  $k$ ,

$$(n \cdot \hat{L}) J |n; j^-, j^+, k, m\rangle = -J (n \cdot \hat{L}) |n; j^-, j^+, k, m\rangle = -m J |n; j^-, j^+, k, m\rangle \quad (\text{C.3})$$

whence

$$J |n; j^-, j^+, k, m\rangle = e^{i\theta_m} |n; j^-, j^+, k, -m\rangle \quad (\text{C.4})$$

for some  $\{\theta_m\} \subset \frac{\mathbb{R}}{2\pi\mathbb{Z}}$ , so that, for all  $m$ ,

$$\begin{aligned} P_S(v \cdot \hat{L})J|n; j^-, j^+, k, m\rangle &= e^{i\theta_m} P_S(v \cdot \hat{L})|n; j^-, j^+, k, -m\rangle \\ &= \chi_S(-\lambda m)J|n; j^-, j^+, k, m\rangle = JP_S(-v \cdot \hat{L})|n; j^-, j^+, k, m\rangle. \end{aligned}$$

■

**Theorem 10.** *The vertex amplitude (4.4) can also be written with a projector on the left instead of on the right:*

$$A_v^{(\Pi+)}(\{k_{ab}, \psi_{ab}\}) := \int_{\text{Spin}(4)^5} \prod_a dG_a \prod_{a < b} \epsilon(\iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} P_{ab}(\{G_{a'b'}\}) \psi_{ab}, \rho(G_{ab}) \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} \psi_{ba}). \quad (\text{C.5})$$

**Proof.** Starting from (4.5), and using lemma 7.c, lemma 8, the hermicity of orthogonal projectors, and lemma 11 in succession,

$$\begin{aligned} A_v^{(\Pi+)}(\{k_{ab}, \psi_{ab}\}) &:= \int_{\text{Spin}(4)^5} \prod_a dG_a \prod_{a < b} \langle J \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} \psi_{ab}, \rho(G_{ab}) \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} P_{ab}(\{G_{a'b'}\}) \psi_{ba} \rangle \\ &= \int_{\text{Spin}(4)^5} \prod_a dG_a \prod_{a < b} \langle J \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} P_{ba}(\{G_{a'b'}\}) \psi_{ab}, \rho(G_{ab}) \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} \psi_{ba} \rangle \end{aligned} \quad (\text{C.6})$$

■

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